that the sum of the terms with n digits in the denominator is greater than $(9/10)^{n-2} \cdot a_2'$, where

$$a_2' = \frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{19} + \frac{1}{21} + \frac{1}{22} + \cdots + \frac{1}{29} + \cdots + \frac{1}{81} + \frac{1}{82} + \cdots + \frac{1}{89};$$

and, therefore, the series is greater than

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + a_2 + [9/10 + (9/10)^2 + \cdots]a_2',$$

that is, greater than

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + a_2 + 9a_2',$$

which turns out to be greater than 22.4.

A still closer approximation may be found by starting with a_3 and a_3', that is, with the terms having three digits in the denominator and with

$$a_2' = \frac{1}{101} + \cdots + \frac{1}{109} + \frac{1}{111} + \cdots + \frac{1}{119} + \cdots + \frac{1}{881} + \cdots + \frac{1}{889}.$$

ON THE MATRIX EQUATION $BX = C$.[1]

By H. T. BURGESS, University of Wisconsin.

Section 1. To Find the Matrix X. The problem is to calculate the elements of the matrix X to satisfy the matrix equation $BX = C$:

$$\begin{Vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{Vmatrix} \begin{Vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{Vmatrix} = \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{Vmatrix}.$$

If we compute the matrix product BX, we get

$$BX = \begin{Vmatrix} \Sigma b_{1\epsilon}x_{\epsilon 1} & \Sigma b_{1\epsilon}x_{\epsilon 2} & \cdots & \Sigma b_{1\epsilon}x_{\epsilon n} \\ \Sigma b_{2\epsilon}x_{\epsilon 1} & \Sigma b_{2\epsilon}x_{\epsilon 2} & \cdots & \Sigma b_{2\epsilon}x_{\epsilon n} \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma b_{n\epsilon}x_{\epsilon 1} & \Sigma b_{n\epsilon}x_{\epsilon 2} & \cdots & \Sigma b_{n\epsilon}x_{\epsilon n} \end{Vmatrix},$$

where the summation runs for $\epsilon = 1, 2, \cdots, n$.

The conditions to be fulfilled are obtained by setting the elements of the product BX equal to the corresponding elements of C. Taking these by columns we get the following n-sets of simultaneous linear equations:

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[1] For the elementary properties of matrices the reader may conveniently consult Bôcher's *Introduction to Higher Algebra*, using the index to find the appropriate sections.

$$\text{I.}\qquad \begin{aligned} b_{11}x_{1\kappa} + b_{12}x_{2\kappa} + \cdots b_{1n}x_{n\kappa} &= c_{1\kappa}, \\ b_{21}x_{1\kappa} + b_{22}x_{2\kappa} + \cdots b_{2n}x_{n\kappa} &= c_{2\kappa}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \quad\cdot\quad\cdot \\ b_{n1}x_{1\kappa} + b_{n2}x_{2\kappa} + \cdots b_{nn}x_{n\kappa} &= c_{n\kappa}, \end{aligned} \qquad \kappa = 1, 2, \cdots, n.$$

The matrix B on the n-sets of unknowns is the same for each of the n-sets of equations, but the column of c's is different for each set. These n-sets may all be solved at once by the following simple device: Write out the two matrices B and C in juxtaposition as one matrix in the form

$$\begin{Vmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} & c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} & c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} & c_{31} & c_{32} & c_{33} & \cdots & c_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} & c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{Vmatrix}.$$

The following operations may be performed on this matrix which give an equivalent matrix in the sense that the n-sets of equations written down from the resulting matrix will have the same solutions as the systems I.

(1) Any two rows may be interchanged,

(2) Any row may be multiplied by a constant not zero,

(3) Any row may be multipled by a constant and added to any other row.

By (1) the element b_{11} can be made different from zero, by (2) it can then be made unity. Next by (3) with $-b_{21}$, $-b_{31}$, $\cdots$, $-b_{n1}$, as multipliers of the first row, all the remaining elements of the first column can be replaced by zeros.

If the matrix B is non-singular, this process can be continued with the successive columns of the matrix until the matrix B is reduced to the unit matrix I and the matrix C is simultaneously reduced to X. For when B is reduced to I, the n-sets of equations have the form

$$\text{II.}\qquad \begin{aligned} x_{1\kappa} \qquad\qquad\qquad &= \bar{c}_{1\kappa}, \\ x_{2\kappa} \qquad\qquad &= \bar{c}_{2\kappa}, \\ x_{3\kappa} \qquad &= \bar{c}_{3\kappa}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \quad\cdot\quad\cdot \\ x_{n\kappa} &= \bar{c}_{n\kappa}, \end{aligned} \qquad \kappa = 1, 2, \cdots, n.$$

Illustration: To determine the matrix X to satisfy the equation

$$\begin{Vmatrix} 1 & 5 & 1 \\ 3 & 4 & 2 \\ 2 & 1 & 3 \end{Vmatrix} \begin{Vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{Vmatrix} = \begin{Vmatrix} 18 & 11 & 3 \\ 20 & 11 & 7 \\ 10 & 4 & 8 \end{Vmatrix}$$

we write

$$\begin{Vmatrix} 1 & 5 & 1 & 18 & 11 & 3 \\ 3 & 4 & 2 & 20 & 11 & 7 \\ 2 & 1 & 3 & 10 & 4 & 8 \end{Vmatrix}.$$

Reducing the first column by (3), we get

$$\begin{Vmatrix} 1 & 5 & 1 & 18 & 11 & 3 \\ 0 & -11 & -1 & -34 & -22 & -2 \\ 0 & -9 & 1 & -26 & -18 & 2 \end{Vmatrix}.$$

Reducing the third column by (3), we get

$$\begin{Vmatrix} 1 & 14 & 0 & 44 & 29 & 1 \\ 0 & -20 & 0 & -60 & -40 & 0 \\ 0 & -9 & 1 & -26 & -18 & 2 \end{Vmatrix}.$$

Dividing the second row by -20 by use of (2), and reducing the second column by (3), we get

$$\begin{Vmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{Vmatrix}. \quad \text{Hence } X = \begin{Vmatrix} 2 & 1 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{Vmatrix}.$$

The method applies to $XB = C$ by use of the conjugates, for $B'X' = C'$.

Section 2. To Find the Inverse of a Matrix. A very useful application occurs when C is replaced by the unit matrix I, for in this case X becomes the inverse of B. The amount of work required to calculate the inverse of a matrix by this method is practically the same as that required to compute one of its elements by the ordinary method.

Illustration: To compute the inverse of the matrix

$$B = \begin{Vmatrix} 1 & -1 & -5 \\ -1 & 3 & -10 \\ -1 & 0 & 12 \end{Vmatrix} \text{ we write } \begin{Vmatrix} 1 & -1 & -5 & 1 & 0 & 0 \\ -1 & 3 & -10 & 0 & 1 & 0 \\ -1 & 0 & 12 & 0 & 0 & 1 \end{Vmatrix}.$$

Reducing column one by (3):

$$\begin{Vmatrix} 1 & -1 & -5 & 1 & 0 & 0 \\ 0 & 2 & -15 & 1 & 1 & 0 \\ 0 & -1 & 7 & 1 & 0 & 1 \end{Vmatrix}.$$

Interchanging rows two and three by (1), changing signs in row two by (2), and reducing column two by (3):

$$\left\|\begin{matrix} 1 & 0 & -12 & 0 & 0 & -1 \\ 0 & 1 & -7 & -1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 1 & 2 \end{matrix}\right\|.$$

Changing the signs in the last row by (2) and reducing column three by (3):

$$\left\|\begin{matrix} 1 & 0 & 0 & -36 & -12 & -25 \\ 0 & 1 & 0 & -22 & -7 & -15 \\ 0 & 0 & 1 & -3 & -1 & -2 \end{matrix}\right\|.$$

Hence,

$$B^{-1} = \left\|\begin{matrix} -36 & -12 & -25 \\ -22 & -7 & -15 \\ -3 & -1 & -2 \end{matrix}\right\|.$$

CENTERS OF SIMILITUDE AND THEIR N-DIMENSIONAL ANALOGIES.

By BANCROFT HUNTINGTON BROWN, Brown University.

1. In the January, 1915, issue of the MONTHLY[1] "Centers of similitude of circles and certain theorems attributed to Monge" were discussed. The theorems there given are the following:

(A) *The six centers of similitude of three coplanar circles lie by threes on four straight lines.*[2]

(B) *The vertices of the six common tangent cones of three spheres, taken in pairs, lie by threes on four straight lines.*

(C) *Given any four spheres in space fixed in magnitude and position, and the six cones tangent to them in pairs, externally; then the six vertices lie in a plane and indeed on four straight lines in the plane. If the six other tangent cones be drawn, then their vertices lie by threes in planes*[3] *with threes of the first group.*

It was shown:

(1) That theorem (A) was, in all probability, known to the Greeks of two thousand years ago;

(2) That Fuss found, with regard to the *external* centers of similitude of coplanar circles, that:

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[1] R. C. Archibald, THE AMERICAN MATHEMATICAL MONTHLY, Vol. XXII, pp. 6–12.

[2] Symbolically, if $E_{m,n}$ denote the external, and $I_{m,n}$ the internal centers of similitude of the circles C_m and C_n $(m, n = 1, 2, 3.\ \ m \gtreqless n)$, the following groups of points are collinear:

$$E_{1,2},\ E_{1,3},\ E_{2,3};\ E_{1,2},\ I_{1,3},\ I_{2,3};\ I_{1,2},\ E_{1,3},\ I_{2,3};\ I_{1,2},\ I_{1,3},\ E_{2,3}.$$

[3] These planes have been called "planes of similitude."